The Laplace Transform

The Laplace transform is used to solve linear constant coefficient differential equations. This is achieved by transforming them to algebraic equations. The algebraic equations are solved, then the inverse Laplace transform is used to obtain a solution in terms of the original variables.

The Laplace transform is also used to produce transfer functions for the elements of an engineering system. Transfer functions are useful in many area of engineering, but are particularly important in the design of control system.

Definition of the Laplace transform

Let $f(t)$ be a function of time t. In many real problems only values of $t \geq 0$ are of interest. Hence $f(t)$ is given for $t \geq 0$, and for all $t < 0$, $f(t)$ is taken to be zero. The Laplace transform of $f(t)$ is $F(s)$, defined by

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

__EX1__: Find the Laplace transform of

a- 1

b- e^{-at}

__Solution__:

a- $$\mathcal{L}\{1\} = \int_0^\infty e^{-st} \cdot 1\, dt = \left[\frac{e^{-st}}{-s}\right]_0^\infty = -\frac{1}{s}\left[\frac{1}{e^{st}}\right]_0^\infty = \frac{1}{s} = F(s)$$

b- $$\mathcal{L}\{e^{-at}\} = \int_0^\infty e^{-st} \cdot e^{-at} dt = \int e^{-(s+a)t} dt = -\frac{1}{s+a} \left. e^{-(s+a)} \right]_0^\infty = \frac{1}{s+a} = F(s)$$

Laplace transforms of some common functions

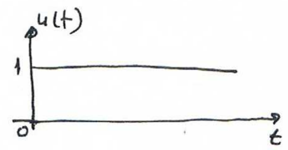
Table below lists some common functions and some common functions and their corresponding Laplace transforms.

Function $f(t)$	Laplace transform $F(s)$
1	$\frac{1}{s}$
t	$1/s^2$
t^2	$2/s^3$
t^n	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
e^{-at}	$\frac{1}{s+a}$
$t^n e^{-at}$	$\frac{n!}{(s+a)^{n+1}}$
$\sin bt$	$\frac{b}{s^2+b^2}$
$\cos bt$	$\frac{s}{s^2+b^2}$

function $f(t)$	Laplace transform $F(s)$
$e^{-at}\sin bt$	$\frac{b}{(s+a)^2+b^2}$
$e^{-at}\cos bt$	$\frac{(s+a)}{(s+a)^2+b^2}$
$t\cdot\sin bt$	$\frac{2bs}{(s^2+b^2)^2}$
$t\cdot\cos bt$	$\frac{s^2-b^2}{(s^2+b^2)^2}$
$u(t)$ unit step	$1/s$
$u(t-d)$	e^{-sd}/s
$\delta(t)$	1
$\delta(t-d)$	e^{-sd}
$\sinh(bt)$	$\frac{b}{s^2-b^2}$
$\cosh(bt)$	$\frac{s}{s^2-b^2}$

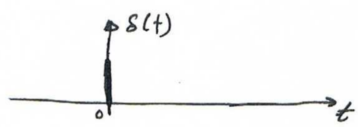
Where $u(t)$ unit step function

$$u(t)=\begin{cases}1 & t\geq 0\\ 0 & t<0\end{cases}$$



$\delta(t)$ is delta function

$$\delta(t)=\begin{cases}1 & t=0\\ 0 & \text{elsewhere}\end{cases}$$



Ex: use table to determine the laplace transform of each of the following functions.

a- t^3 b- t^7 c- $\sin 4t$ d- e^{-2t} e- $\cos(\frac{t}{2})$ f- $t \cdot \sin 4t$

g- $e^{-t}\sin 2t$ h- $e^{3t}\cos t$ i- $\sinh 6t$ j- $\cosh 5t$ k- $e^{7t}\cosh 9t$

Solution

a. $\mathcal{L}\{t^3\} = \frac{3!}{s^{3+1}} = \frac{6}{s^4} = F(s)$

b- $\mathcal{L}\{t^7\} = \frac{7!}{s^{7+1}} = \frac{7!}{s^8} = F(s)$

c- $\mathcal{L}\{\sin 4t\} = \frac{4}{s^2+4^2} = \frac{4}{s^2+16} = F(s)$

d- $\mathcal{L}\{e^{-2t}\} = \frac{1}{s-(-2)} = \frac{1}{s+2} = F(s)$

e. $\mathcal{L}\{\cos(\frac{t}{2})\} = \frac{s}{s^2+(\frac{1}{2})^2} = \frac{s}{s^2+0.25} = F(s)$

f- $\mathcal{L}\{t \cdot \sin 4t\} = \frac{8s}{(s^2+4^2)^2} = \frac{8s}{(s^2+16)^2} = F(s)$

g- $\mathcal{L}\{e^{-t} \cdot \sin 2t\} = \frac{2}{(s+1)^2+2^2} = \frac{2}{(s+1)^2+4} = F(s)$

h- $\mathcal{L}\{e^{3t}\cos t\} = \frac{(s-3)}{(s-3)^2+1^2} = \frac{(s-3)}{(s-3)^2+1}$

i- $\mathcal{L}[\sinh 6t\} = \frac{6}{s^2-36}$

j- $\mathcal{L}[\cosh 5t) = \frac{s}{s^2-25}$

k- $\mathcal{L}[e^{7t}\cosh 9t] = \frac{s-7}{(s-7)^2 - 81}$

Exercises

- Determine the laplace transforms of the following functions.

a- $\sin 6t$ b- $\cos 4t$ c- $\sin\left(\frac{2t}{3}\right)$ d- $\cos\left(\frac{4t}{3}\right)$ e- t^4

f- $t^2 \cdot t^3$ g- e^{-3t} h- e^{3t} i- $\frac{1}{e^{4t}}$ j- $t \cdot \cos 3t$ k- $t \sin t$

l- $e^{-t} \sin t$ m- $\frac{\cos 7t}{e^{5t}}$

Solutions

a- $\frac{6}{s^2+36}$ b- $\frac{s}{s^2+16}$ c- $\frac{6}{9s^2+4}$ e- $\frac{24}{s^5}$

d- $\frac{9s}{9s^2+16}$ f- $\frac{120}{s^6}$ g- $\frac{1}{s+3}$ h- $\frac{1}{s-3}$

i- $\frac{1}{s+4}$ j- $\frac{s^2-9}{(s^2+9)^2}$ k- $\frac{2s}{(s^2+1)^2}$ l- $\frac{3}{(s+1)^2+9}$

m- $\frac{s+5}{(s+5)^2+49}$

Rule

Let $\mathcal{L}\{f(t)\} = F(s)$, then

$$\boxed{\mathcal{L}\{t^n \cdot f(t)\} = \frac{(-1)^n d^n}{ds^n}\left[F(s)\right]}$$

Ex: Find the Laplace transform of the following

1- $t \cdot \cos 2t$ 2- $t^2 \cdot e^{-t}$ 3- $t^2 \cdot \sin 3t$

4- $t \cosh(4t)$

Solution

1- $\mathcal{L}\{t \cdot \cos 2t\} \Rightarrow f(t) = \cos 2t$, $n = 1$

$$F(s) = \mathcal{L}\{f(t)\} = \mathcal{L}\{\cos 2t\} = \frac{s}{s^2+4}$$

$$\mathcal{L}\{t \cdot \cos 2t\} = (-1)^1 \cdot \frac{d}{ds}\left[\frac{s}{s^2+4}\right] = -1 \cdot \left[\frac{(s^2+4) - s \cdot (2s)}{(s^2+4)^2}\right]$$

$$= -1\left[\frac{s^2+4-2s^2}{(s^2+4)^2}\right] = -1\left[\frac{4-s^2}{(s^2+4)^2}\right]$$

$$= \frac{s^2-4}{(s^2+4)^2}$$

2. $\mathcal{L}\{t^2 \cdot e^{-t}\} \Rightarrow f(t) = e^{-t}$, $n = 2$, $F(s) = \frac{1}{s+1}$

$$\mathcal{L}\{t^2 \cdot e^{-t}\} = (-1)^n \cdot \frac{d^2}{ds^2}\left[\frac{1}{s+1}\right] = \frac{d}{ds}\left[\frac{-1}{(s+1)^2}\right]$$

$$= \frac{2\cancel{(s+1)}}{(s+1)^{\cancel{4}\,3}} = \frac{2}{(s+1)^3}$$

3- $\mathcal{L}\{t^2 \cdot \sin 3t\}$, $f(t) = \sin 3t$, $n=2$

$$F(s) = \mathcal{L}\{f(t)\} = \frac{3}{s^2+9}$$

$$\mathcal{L}\{t^2 \cdot \sin 3t\} = (-1)^2 \cdot \frac{d^2}{ds^2}\left[\frac{3}{s^2+9}\right]$$

$$= \frac{d}{ds}\left[\frac{-3(2s)}{(s^2+9)^2}\right] = \frac{d}{ds}\left[\frac{-6s}{(s^2+9)^2}\right]$$

$$= \frac{-6[s^2+9]^2 - (-6s)\cdot 2(s^2+9)\cdot 2s}{(s^2+9)^4}$$

$$= \frac{-6[s^2+9]^2 + 24s^2(s^2+9)}{(s^2+9)^4} = \frac{\cancel{(s^2+9)}[24s^2 - 6(s^2+9)]}{(s^2+9)^{\cancel{4}\,3}}$$

$$= \frac{24s^2 - 6(s^2+9)}{(s^2+9)^3} = \frac{18s^2 - 54}{(s^2+9)^3}$$

4- $\mathcal{L}\{t \cdot \cosh 4t\}$, $f(t) = \cosh 4t$, $n=1$

$$= (-1)^n \frac{d^n}{ds^n} F(s) = -1 \frac{d}{ds} \{F(s)\}$$

$$F(s) = \mathcal{L}\{f(t)\} = \frac{s}{s^2-16}$$

$$\mathcal{L}\{t \cdot \cosh 4t\} = -\frac{d}{ds}\left[\frac{s}{s^2-16}\right] = -\frac{s^2-16-s(2s)}{(s^2-16)^2} = -\frac{s^2+16}{(s^2-16)^2}$$

Properties of the Laplace transform

There are some useful properties of the laplace transform that can be exploited. They allow us to find the laplace transforms of more difficult functions. The properties we shall examine are:

1- Linearity.

2- shift theorems.

3- Final value theorem.

1- Linearity

Let f and g be two functions of t and let K be a constant which may be negative. Then

$$\mathcal{L}\{f+g\} = \mathcal{L}\{f\} + \mathcal{L}\{g\}$$

$$\mathcal{L}\{Kf\} = K\,\mathcal{L}\{f\}$$

Example: Find the laplace transforms of the following functions:

(a) $3+2t$ (b) $5t^2-2e^t$

Solution

(a) $\mathcal{L}\{3+2t\} = \mathcal{L}\{3\} + \mathcal{L}\{2t\} = 3\mathcal{L}\{1\} + 2\mathcal{L}\{t\}$

$= \frac{3}{s} + \frac{2}{s^2}$

(b) $\mathcal{L}\{5t^2-2e^t\} = \mathcal{L}\{5t^2\} + \mathcal{L}\{-2e^t\} = \frac{10}{s^3} - \frac{2}{s-1}$

Ex: Find the Laplace transforms of the following:

(a) $5\cos 3t + 2\sin 5t - 6t^3$

(b) $-e^{-t} + \frac{1}{2}(\sin t + \cos t)$

Solution

a. $\mathcal{L}\{5\cos 3t + 2\sin 5t - 6t^3\} = \frac{5s}{s^2+9} + \frac{10}{s^2+25} - \frac{36}{s^4}$

b. $\mathcal{L}\left\{-e^{-t} + \frac{\sin t + \cos t}{2}\right\} = \frac{-1}{s+1} + \frac{0.5}{s^2+1} + \frac{0.5s}{s^2+1}$

$$= -\frac{1}{s+1} + \frac{0.5(s+1)}{s^2+1}$$

2. Shift theorem

2.1 First shift theorem

If $\mathcal{L}\{f(t)\} = F(s)$ then

$\mathcal{L}\{e^{-at} f(t)\} = F(s+a)$ $\quad a = \text{constant}$

We obtain $F(s+a)$ by replacing every $\{s\}$ in $F(s)$ by $(s+a)$. The variable s has been shifted by an amount a.

Ex: a- Find the laplace transform of

$f(t) = t \cdot \sin 5t$

b- use the first shift theorem to write down

$\mathcal{L}\{e^{-3t} t \sin 5t\}$

Solution (a)

(a) $\mathcal{L}\{t \cdot \sin 5t\} = (-1)\frac{d}{ds}\left\{\frac{5}{s^2+25}\right\} = -\frac{-10s}{(s^2+25)^2}$

$= \frac{10s}{(s^2+25)^2} = F(s)$

(b) $\mathcal{L}\{e^{-3t} t \cdot \sin 5t\} = F(s)\Big/_{s=s+3} = F(s+3)$

$= \frac{10(s+3)}{\left((s+3)^2+25\right)^2} = \frac{10(s+3)}{(s^2+6s+34)^2}$

Ex: The laplace transform of a function, $f(t)$, is given by

$F(s) = \frac{2s+1}{s(s+1)}$

state the Laplace transform of

(a) $e^{-2t} f(t)$ (b) $e^{3t} f(t)$

Solution

(a) $\mathcal{L}\{e^{-2t}f(t)\} = F(s+2) = \dfrac{2(s+2)+1}{(s+2)(s+2+1)}$

$= \dfrac{2s+5}{(s+2)(s+3)}$

(b) $\mathcal{L}\{e^{3t}f(t)\} = F(s-3) = \dfrac{2(s-3)+1}{(s-3)(s-3+1)}$

$= \dfrac{2s-5}{(s-3)(s-2)}$

2.2 Second shift theorem

If $\mathcal{L}\{f(t)\} = F(s)$ then

$$\mathcal{L}\{u(t-d)\cdot f(t-d)\} = e^{-sd}F(s)$$

The function $[u(t-d)\cdot f(t-d)]$ is obtained by moving $[u(t)\cdot f(t)]$ to right by an amount d. Note that because $f(t)$ is defined to be zero for $t<0$, the $f(t-d)=0$ for $t<d$.

Ex: Given $\mathcal{L}\{f(t)\} = \dfrac{2s}{s+9}$, find $\mathcal{L}\{u(t-2)f(t-2)\}$.

Solution $\mathcal{L}\{u(t-2)f(t-2)\} = e^{-sd}F(s) = e^{-2s}\cdot\dfrac{2s}{s+9}$

$= \dfrac{2se^{-2s}}{s+9}$

Ex: The laplace transform of a function is $\frac{e^{-3s}}{s^2}$. find the function

<u>solution</u>

$$\mathcal{L}\{u(t-d)\cdot f(t-d)\} = e^{-ds}F(s)$$

$$e^{-3s}\cdot\frac{1}{s^2} = e^{-ds}F(s)$$

$$\therefore d=3,\quad F(s)=\frac{1}{s^2}$$

$$\therefore F(s)=\mathcal{L}\{f(t)\}=\mathcal{L}\{t\}=\frac{1}{s^2}$$

$$\therefore f(t)=t$$

$$\therefore \mathcal{L}\{u(t-3).f(t-3)\}=\frac{e^{-3s}}{s^2}$$

Ex: Find the laplace transform of the following

$$\mathcal{L}\{u(t-4).\sin(t-4)\},\ \text{if } f(t)=\sin t$$

<u>solution</u> $\mathcal{L}\{f(t)\}=F(s)=\frac{1}{s^2+1}$, $d=4$

$$\mathcal{L}\{u(t-4).\sin(t-4)\}=e^{-ds}.F(s)=\frac{e^{-4s}}{s^2+1}$$

3- Final value theorem

The final value theorem states

$$\lim_{s \to 0} sF(s) = \lim_{t \to \infty} f(t)$$

Ex: verify the final value theorem for $f(t) = e^{-2t}$

Solution $\mathcal{L}\{f(t)\} = F(s) = \mathcal{L}\{e^{-2t}\} = \frac{1}{s+2}$

$$\lim_{s \to 0} s \cdot F(s) = \lim_{s \to 0} \frac{s}{s+2} = \lim_{s \to 0} \frac{1}{1+\frac{2}{s}} = 0$$

$$\lim_{t \to \infty} f(t) = \lim_{t \to \infty} e^{-2t} = \lim_{t \to \infty} \frac{1}{e^{2t}} = \frac{1}{\infty} = 0$$

$$\therefore \lim_{s \to 0} sF(s) = \lim_{t \to \infty} f(t)$$

Exercises

1- Find the Laplace transforms of the following functions.

(a) $3t^2-4$ (b) $2\sin 4t+11-t$ (c) $3e^{2t}+4\sin t$

(d) $e^{-t}\sin 3t+4e^{-t}\cos 3t$ (e) $3t^4e^{5t}+t$

2. The laplace transform is given as $F(s)=\dfrac{3s^2-1}{s^2+s+1}$

Find the laplace transform of

(a) $e^{-t}f(t)$ (b) $e^{3t}f(t)$ (c) $e^{-t/2}f(t)$

3. Given $\mathcal{L}\{f(t)\}=\dfrac{4s}{s^2+1}$

Find

(a) $\mathcal{L}\{u(t-1)f(t-1)\}$ (b) $\mathcal{L}\{3u(t-2)f(t-2)\}$

(c) $\mathcal{L}\left\{u(t-4)\cdot\dfrac{f(t-4)}{2}\right\}$

4. Find the final value of the following functions using the final value theorem:

(a) $f(t)=e^{-t}\sin t$ (b) $f(t)=e^{-t}+1$

(c) $f(t)=e^{-3t}\cos t+5$

Solutions

1- a) $\dfrac{6}{s^3} - \dfrac{4}{s}$ b) $\dfrac{8}{s^2+16} + \dfrac{11}{s} - \dfrac{1}{s^2}$

c) $\dfrac{3}{s-2} + \dfrac{4}{s^2+1}$ d) $\dfrac{4s+7}{(s+1)^2+9}$ e) $\dfrac{72}{(s-5)^2} + \dfrac{1}{s^2}$

2. a) $\dfrac{3s^2+6s+2}{s^2+3s+3}$ b) $\dfrac{3s^2-18s+26}{s^2-5s+7}$ c) $\dfrac{12s^2+12s-1}{4s^2+8s+7}$

3- a) $\dfrac{4s e^{-s}}{s^2+1}$ b) $\dfrac{12s e^{-2s}}{s^2+1}$ c) $\dfrac{2s e^{-4s}}{s^2+1}$

4- a) 0 b) 1 c) 5

Laplace transform of derivatives and integrals

Let $f(t)$ be a function of t, and f' and f'' the first and second derivatives of f. The Laplace transform of $f(t)$ is $F(s)$. Then

$$\mathcal{L}\{f'\} = sF(s) - f(0)$$

$$\mathcal{L}\{f''\} = s^2F(s) - sf(0) - f'(0)$$

Where $f(0)$ and $f'(0)$ are the initial values of f and f'. The general case for the Lapalace transform of an nth derivative is

$$\mathcal{L}\{f^{(n)}\} = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{n-1}(0)$$

Another useful result is

$$\mathcal{L}\left\{\int_0^t f(t)\,dt\right\} = \frac{1}{s}F(s)$$

Example: The Laplace transform of $x(t)$ is $X(s)$. Given $x(0)=2$ and $\dot{x}(0)=-1$, write expressions for the Laplace transform of

ⓐ $2x'' - 3x' + x$

ⓑ $-x'' + 2x' + x$

Solution

$$\mathcal{L}\{\dot{x}\} = sX(s) - x(0) = sX(s) - 2$$

$$\mathcal{L}\{x''\} = s^2X(s) - sx(0) - \dot{x}(0) = s^2X(s) - 2s + 1$$

ⓐ $\mathcal{L}\{2x''-3x'+x\} = 2(s^2X(s)-2s+1)-3(sX(s))+6+X(s)$

$= (2s^2-3s+1)X(s)-4s+8$

ⓑ $\mathcal{L}\{-x''+2x'+x\} = -(s^2X(s)-2s+1)+2(sX(s))+X(s)$

$= (-s^2+2s+1)X(s)+2s-5$

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Exercises

1- The laplace transform of $y(t)$ is $Y(s)$, $y(0)=3$, $\dot{y}(0)=1$.
Find the laplace transform of the following expressions.

ⓐ- y' ⓑ- $3y''-y'$ ⓒ- $3\frac{d^2y}{dt^2}+6\frac{dy}{dt}+8y$

2- Given the Laplace transform of $f(t)$ is $F(s)$, $f(0)=2$, $\dot{f}(0)=3$ and $f''(0)=-1$. Find the Laplace transforms of

ⓐ- f''' ⓑ- $2f'''-f''+4f'-2f$

Solutions

1- ⓐ $sY(s)-3$ ⓑ- $3s^2Y(s)-sY(s)-9s$

ⓒ $(3s^2+6s+8)Y(s)-9s-21$

2- ⓐ $s^3F(s)-2s^2-3s+1$

ⓑ $(2s^3-s^2+4s-2)F(s)-4s^2-4s-3$

Inverse Laplace transforms

The laplace transform can be used to solve differential equations. However, before such an application can be put into practice, we must study the inverse Laplace transform. If $\mathcal{L}\{f(t)\} = F(s)$ we write

$$f(t) = \mathcal{L}^{-1}\{F(s)\}$$

and call $\mathcal{L}^{-1}$ the inverse Laplace transform. like $\mathcal{L}$, $\mathcal{L}^{-1}$ can be shown to be a linear operator.

Ex: Find the inverse Laplace transforms of the following:

a- $\frac{2}{s^3}$ b- $\frac{16}{s^3}$ c- $\frac{s}{s^2+1}$ d- $\frac{1}{s^2+1}$ e- $\frac{s+1}{s^2+1}$

f- $\frac{10}{(s+2)^4}$ g- $\frac{(s+1)}{(s+1)^2+4}$ h- $\frac{s+3}{s^2+6s+13}$ i- $\frac{2s+3}{s^2+6s+13}$

Solution

a- $\mathcal{L}^{-1}\left\{\frac{2}{s^3}\right\} = t^2$

b- $\mathcal{L}^{-1}\frac{16}{s^3} = \mathcal{L}^{-1}\frac{8\times 2}{s^3} = 8\mathcal{L}^{-1}\frac{2}{s^3} = 8t^2$

c- $\mathcal{L}^{-1}\frac{s}{s^2+1} = \cos t$

d- $\mathcal{L}^{-1}\frac{1}{s^2+1} = \sin t$

e- $\mathcal{L}^{-1}\frac{(s+1)}{s^2+1} = \mathcal{L}^{-1}\left(\frac{s}{s^2+1} + \frac{1}{s^2+1}\right) = \mathcal{L}^{-1}\frac{s}{s^2+1} + \mathcal{L}^{-1}\frac{1}{s^2+1}$

$= \cos t + \sin t$

f- $\mathcal{L}^{-1}\frac{10}{(s+2)^4} = 10\mathcal{L}^{-1}\frac{1}{(s+2)^4}\cdot\frac{6}{6} = \frac{10}{6}\mathcal{L}^{-1}\frac{6}{(s+2)^4}$

$= \frac{10}{6}\,t^3\cdot e^{-2t}$

g- $\mathcal{L}^{-1}\frac{(s+1)}{(s+1)^2+4} = \cos 2t\cdot e^{-t}$

h- $\mathcal{L}^{-1}\frac{s+3}{s^2+6s+13} = \mathcal{L}^{-1}\frac{s+3}{s^2+6s+9+4} = \mathcal{L}^{-1}\frac{s+3}{(s+3)^2+4}$

$= \mathcal{L}^{-1}\frac{(s+3)}{(s+3)^2+(2)^2} = \cos 2t\cdot e^{-3t}$

i- $\mathcal{L}^{-1}\frac{2s+3}{s^2+6s+13} = \mathcal{L}^{-1}\frac{2s+3\ \underline{+3-3}}{(s+3)^2+(2)^2} = \mathcal{L}^{-1}\frac{2s+6-3}{(s+3)^2+(2)^2}$

$= \mathcal{L}^{-1}\frac{2s+6}{(s+3)^2+(2)^2} - \mathcal{L}^{-1}\frac{3}{(s+3)^2+(2)^2}$

$= 2\,\mathcal{L}^{-1}\frac{s+3}{(s+3)^2+(2)^2} - \mathcal{L}^{-1}\frac{3}{(s+3)^2+(2)^2}\left(\frac{2}{2}\right)$

$= 2\cos 2t\cdot e^{-3t} - \frac{3}{2}\sin 2t\cdot e^{-3t}$

Exercises

Find the inverse Laplace transforms of the following functions:

a- $\frac{1}{s}$ b- $\frac{2}{s^3}$ c- $\frac{8}{s}$ d- $\frac{1}{s+2}$ e- $\frac{6}{s+4}$ f- $\frac{5}{s-3}$

g- $\frac{7}{s+8}$ h- $\frac{2}{s^2+4}$ i- $\frac{s}{s^2+9}$ j- $\frac{6s}{s^2+4}$ k- $\frac{20}{s^2+16}$

l- $\frac{2}{(s+1)^2}$ m- $\frac{8}{(s+2)^2}$ n- $\frac{9}{(s+3)^3}$ o- $\frac{e^{-3s}}{s}$ p- $\frac{4e^{-6s}}{s}$

q- e^{-9s} r- $4e^{-8s}$ s- $\frac{4}{s}-\frac{1}{s^3}$ t- $\frac{3s-7}{s^2+9}$

u- $\frac{s-6}{s-4}$ v- $\frac{s+4}{(s+4)^2+1}$ w- $\frac{6s+17}{(s+4)^2+1}$ x- $\frac{s}{s^2+2s+7}$

y- $\frac{s+5}{s^2+8s+20}$ z- $\frac{7s+3}{s^2+4s+8}$

Solutions

a- 1 b- t^2 c- 8 d- e^{-2t} e- $6e^{-4t}$ f- $5e^{3t}$

g- $7e^{-8t}$ h- $\sin 2t$ i- $\cos 3t$ j- $6\cos 2t$ k- $5\sin 4t$

l- $2te^{-t}$, m- $8te^{-2t}$ n- $\frac{9}{2}t^2e^{-3t}$ o- $u(t-3)$ p- $4u(t-6)$

q- $\delta(t-9)$ r- $4\delta(t-8)$ s- $4-\frac{t^2}{2}$ t- $3\cos 3t-\frac{7}{3}\sin 3t$

u- $\delta(t)-2e^{4t}$ v- $e^{-4t}\cos t$ w- $6e^{-4t}\cos t-7e^{-4t}\sin t$

x- $e^{-t}\cos(\sqrt{6}t)-\frac{1}{\sqrt{6}}e^{-t}\cdot\sin(\sqrt{6}t)$ y- $e^{-4t}\cos 2t+\frac{1}{2}e^{-4t}\sin 2t$

z- $7e^{-2t}\cos 2t-\frac{11}{2}e^{-2t}\sin 2t$

Using partial fractions to find the inverse Laplace transform

Ex: Find the inverse Laplace transform of

a- $\dfrac{4s-1}{s^2-s}$ b- $\dfrac{6s+8}{s^2+3s+2}$ c- $\dfrac{3s^2+6s+2}{s^3+3s^2+2s}$

Solution

a- $$F(s) = \frac{4s-1}{s^2-s} = \frac{4s-1}{s(s-1)} = \frac{A}{s} + \frac{B}{s-1}$$

$$A = F(s)\cdot s \Big|_{s=0} = \frac{4s-1}{\cancel{s}(s-1)}\cdot \cancel{s}\Big|_{s=0} = 1$$

$$B - F(s)\cdot(s-1)\Big|_{s-1=0} = \frac{4s-1}{s\cancel{(s-1)}}\cdot\cancel{(s-1)}\Big|_{s=1} = \frac{4-1}{1} = 3$$

$$\therefore F(s) = \frac{A}{s} + \frac{B}{s-1} = \frac{1}{s} + \frac{3}{s-1}$$

$$\therefore \mathcal{L}^{-1}F(s) = \mathcal{L}^{-1}\left(\frac{1}{s} + \frac{3}{s-1}\right) = \mathcal{L}^{-1}\frac{1}{s} + \mathcal{L}^{-1}\frac{3}{s-1}$$

$$= 1 + 3e^{t}$$

b- $F(s) = \dfrac{6s+8}{s^2+3s+2} = \dfrac{6s+8}{(s+1)(s+2)} = \dfrac{A}{s+1} + \dfrac{B}{s+2}$

$$A = F(s)\cdot(s+1)\Big|_{s=-1} = \frac{6s+8}{\cancel{(s+1)}(s+2)}\cdot\cancel{(s+1)}\Big|_{s=-1} = \frac{2}{1} = 2$$

$$B = F(s)\cdot(s+2)\Big|_{s+2=0} = \frac{6s+8\cdot\cancel{(s+2)}}{(s+1)\cancel{(s+2)}}\Big|_{s=-2} = \frac{-4}{-1} = 4$$

$$\therefore\ F(s) = \frac{A}{s+1} + \frac{B}{s+2} = \frac{2}{s+1} + \frac{4}{s+2}$$

$$\mathcal{L}^{-1}F(s) = \mathcal{L}^{-1}\left(\frac{2}{(s+1)} + \frac{4}{(s+2)}\right) = \mathcal{L}^{-1}\frac{2}{(s+1)} + \mathcal{L}^{-1}\frac{4}{(s+2)}$$

$$= 2e^{-t} + 4e^{-2t}$$

c- $F(s) = \dfrac{3s^2+6s+2}{s^3+3s^2+2s} = \dfrac{3s^2+6s+2}{s(s^2+3s+2)} = \dfrac{3s^2+6s+2}{s(s+1)(s+2)}$

$$F(s) = \frac{A}{s} + \frac{B}{(s+1)} + \frac{C}{(s+2)}$$

$A = F(s)\cdot s\Big|_{s=0} = 1$, $B = F(s)\cdot(s+1)\Big|_{s=-1} = 1$, $C = F(s)\cdot(s+2)\Big|_{s=-2} = 1$

$$F(s) = \frac{1}{s} + \frac{1}{(s+1)} + \frac{1}{(s+2)}$$

$$\mathcal{L}^{-1}F(s) = 1 + e^{-t} + e^{-2t}$$

Exercises

Express the following as partial fractions and hence find their inverse Laplace transforms!

a- $\dfrac{5s+2}{(s+1)(s+2)}$ b- $\dfrac{3s-4}{(s+2)(s+3)}$ c- $\dfrac{4s+1}{(s+3)(s+4)}$

d- $\dfrac{6s-5}{(s+5)(s+3)}$ e- $\dfrac{4s+1}{s(s+2)(s+3)}$ f- $\dfrac{7s+3}{s(s+3)(s+4)}$

g- $\dfrac{6s+7}{s(s+2)(s+4)}$ h- $\dfrac{8s-5}{(s+1)(s+2)(s+3)}$ i- $\dfrac{3s+5}{(s+1)(s^2+3s+2)}$

j- $\dfrac{2s-8}{(s+2)(s^2+7s+6)}$ k- $\dfrac{3s+3}{(s-1)(s+2)}$ l- $\dfrac{5s}{(s+1)(2s-1)}$

o- $\dfrac{2s+5}{s+2}$ p- $\dfrac{1-s}{(s+1)(s^2+2s+2)}$ q- $\dfrac{3s^2-s+8}{(s^2-2s+3)(s+2)}$

Solutions

a- $8e^{-2t}-3e^{-t}$ b- $5e^{-3t}-2e^{-2t}$ c- $15e^{-4t}-11e^{-3t}$ d- $\frac{35}{2}e^{-5t}-\frac{23}{2}e^{-3t}$

e- $\frac{1}{6}+\frac{7}{2}e^{-2t}-\frac{11}{3}e^{-3t}$ f- $\frac{1}{4}+6e^{-3t}-\frac{25}{4}e^{-4t}$ g- $\frac{7}{8}+\frac{5}{4}e^{-2t}-\frac{17}{8}e^{-4t}$

h- $21e^{-2t}-\frac{29}{2}e^{-3t}-\frac{13}{2}e^{-t}$ i- $e^{-t}+2te^{-t}-e^{-2t}$ j- $3e^{-2t}-e^{-6t}-2e^{-t}$

k- $2e^{t}+e^{-2t}$ l- $e^{-t}+\frac{3}{2}e^{\frac{3}{2}t}$ o- $2\delta(t)+e^{-2t}$ p- $e^{-t}(2-2\cos t-\sin t)$

q- $e^{t}(\cos(\sqrt{2}t)+\sqrt{2}\sin(\sqrt{2}t))+2e^{-2t}$

Solving linear constant coefficient differential equations using the laplace transform

The laplace transform of the differential equation is found. This transforms the differential equation into an algebraic equation. The transform of the dependent variable is found and then the inverse transform is calculated to yield the required particular solution.

Ex1: solve the differential equation

$$\frac{dx}{dt} + x = 0 \qquad x(0) = 3 \quad \text{using Laplace transforms.}$$

<u>solution</u> $\dot{x} + x = 0$

$$\mathcal{L}\{\dot{x}\} = sX(s) - x(0) = sX(s) - 3$$

$$\mathcal{L}\{\dot{x} + x = 0\} \Rightarrow \mathcal{L}\{\dot{x}\} + \mathcal{L}\{x\} = 0$$

$$sX(s) - 3 + X(s) = 0$$

$$X(s)(s+1) - 3 = 0 \implies X(s)(s+1) = 3$$

$$\therefore X(s) = \frac{3}{s+1}$$

$$x(t) = \mathcal{L}^{-1} X(s) = 3e^{-t}$$

Ex 2 / Solve $\frac{dx}{dt} + x = 9e^{2t}$, $x(0) = 3$

using the Laplace transform.

solution $\dot{x} + x = 9e^{2t}$

$$\mathcal{L}\{\dot{x}\} = sX(s) - X(0) = sX(s) - 3$$

The Laplace transform of both sides of the equation is found.

$$\mathcal{L}\{\dot{x}\} + \mathcal{L}\{x\} = \mathcal{L}\{9e^{2t}\}$$

$$sX(s) - 3 + X(s) = 9\left(\frac{1}{s-2}\right)$$

$$X(s)(s+1) - 3 = \frac{9}{s-2}$$

$$X(s)(s+1) = \frac{9}{s-2} + 3 = \frac{9 + 3(s-2)}{(s-2)}$$

$$X(s)(s+1) = \frac{3s+3}{(s-2)}$$

$$\therefore X(s) = \frac{3s+3}{(s-2)(s+1)} = \frac{A}{(s-2)} + \frac{B}{(s+1)}$$

$$A = X(s)\cdot(s-2)\Big|_{s=2} = \frac{3s+3}{\cancel{(s-2)}(s+1)}\cdot\cancel{(s-2)}\Big|_{s=2} = 3$$

$$B = X(s)\cdot(s+1)\Big|_{s=-1} = 0$$

$$\therefore X(s) = \frac{3}{(s-2)} \implies x(t) = \mathcal{L}^{-1}X(s) = \mathcal{L}^{-1}\left(\frac{3}{(s-2)}\right) = 3e^{2t}$$

EX3: Solve $y'' + y' - 2y = -2$ $\quad y(0) = 2,\ y'(0) = 1$

Solution

$$\mathcal{L}(y'') = s^2 Y(s) - s \cdot y(0) - y'(0) = s^2 Y(s) - 2s - 1$$

$$\mathcal{L}(y') = s Y(s) - y(0) = s Y(s) - 2$$

$$\mathcal{L}(y'') + \mathcal{L}(y') - 2\mathcal{L}(y) = \mathcal{L}(-2)$$

$$\underline{s^2 Y(s)} - 2s - 1 + \underline{s Y(s)} - 2 - \underline{2 Y(s)} = \frac{-2}{s}$$

$$Y(s)\left(s^2 + s - 2\right) - 2s - 3 = \frac{-2}{s}$$

$$Y(s)\left(s^2 + s - 2\right) = \frac{-2}{s} + 2s + 3 = \frac{-2 + 2s^2 + 3s}{s}$$

$$\therefore Y(s) = \frac{2s^2 + 3s - 2}{s(s^2 + s - 2)} = \frac{2s^2 + 3s - 2}{s(s+2)(s-1)} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s-1}$$

$$A = Y(s) \cdot s \Big|_{s=0} = 1, \quad B = Y(s) \cdot (s+2) \Big|_{s=-2} = 0, \quad C = Y(s) \cdot (s-1) \Big|_{s=1} = 1$$

$$\therefore Y(s) = \frac{1}{s} + \frac{1}{(s-1)}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left(\frac{1}{s}\right) + \mathcal{L}^{-1}\left(\frac{1}{s-1}\right)$$

$$y(t) = 1 + e^{t}$$

Ex4: Solve $\ddot{x} - 5\dot{x} + 6x = 6t - 4$ $\quad x(0) = 1,\ \dot{x}(0) = 2$

Solution

$$\mathcal{L}(x'') = s^2X(s) - s\,\cancel{X}(0) - \dot{x}(0) = s^2X(s) - s - 2$$

$$\mathcal{L}\{x'\} = sX(s) - \cancel{X}(0) = sX(s) - 1$$

$$\mathcal{L}\{x''\} - 5\mathcal{L}\{x'\} + 6\mathcal{L}\{x\} = 6\mathcal{L}\{t\} - 4\mathcal{L}\{1\}$$

$$s^2X(s) - s - 2 - 5sX(s) + 5 + 6X(s) = \frac{6}{s^2} - \frac{4}{s}$$

$$X(s)(s^2 - 5s + 6) - s + 3 = \frac{6}{s^2} - \frac{4}{s}$$

$$X(s)(s^2 - 5s + 6) = \frac{6}{s^2} - \frac{4}{s} + s - 3 = \frac{6 - 4s + s^3 - 3s^2}{s^2}$$

$$X(s) = \frac{s^3 - 3s^2 - 4s + 6}{s^2(s^2 - 5s + 6)} = \frac{s^3 - 3s^2 - 4s + 6}{s^2(s-2)(s-3)}$$

$$X(s) = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{(s-2)} + \frac{D}{(s-3)}$$

$$A = X(s)\cdot s^2\Big|_{s=0} = 1$$

$$B = \frac{d}{ds}\left[X(s)\cdot s^2\right]\Big|_{s=0} = \frac{d}{ds}\left[\frac{s^3 - 3s^2 - 4s + 6}{(s-2)(s-3)}\right]\Bigg|_{s=0} = \frac{(s-2)(s-3)(3s^2 - 6s - 4) - (s^3 - 3s^2 - 4s + 6)(2s-5)}{[(s-2)(s-3)]^2}$$

$$B = \frac{-24 - (-30)}{36} = \frac{6}{36} = \frac{1}{6}$$

$$C = X(s)\cdot(s-2)\Big|_{s=2} = \frac{8 - 12 - 8 + 6}{-4} = \frac{-6}{-4} = 3/2$$

$$D = X(s)\cdot(s-3)\Big|_{s=3} = \frac{27 - 27 - 12 + 6}{9} = \frac{-6}{9} = -\frac{2}{3}$$

$$\therefore X(s) = \frac{1}{s^2} + \frac{1/6}{s} + \frac{3/2}{(s-2)} + \frac{-2/3}{(s-3)}$$

$$\therefore \mathcal{L}^{-1}X(s) = \mathcal{L}^{-1}\left(\frac{1}{s^2}\right) + \frac{1}{6}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} - \frac{2}{3}\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\}$$

$$x(t) = t + \frac{1}{6} + \frac{2}{3}e^{2t} - \frac{2}{3}e^{3t}$$

Exercises

use lapalce transform to solve

a- $\dot{x} + x = 3$, $x(0) = 1$

b- $\ddot{x} + \dot{x} - 2x = 1 - 2t$
$x(0) = 6$, $\dot{x}(0) = -11$

c- $2\frac{dy}{dx} + 4y = 1$, $y(0) = 4$

d- $\ddot{x} + x = 2t$
$x(0) = 0$, $\dot{x}(0) = 5$

solutions

a- $x(t) = 3 - 2e^{-t}$

b- $x(t) = t + 6e^{-2t}$

c- $y(t) = \frac{1}{4} + \frac{15}{4}e^{-2t}$

d- $x(t) = 3\sin t + 2t$

partial Fraction for complex Roots:

If there are complex roots, then $F(s)$ can be written as,

$$\boxed{F(s) = \frac{N(s)}{(s-\alpha-j\beta)(s-\alpha+j\beta)\, D_1(s)}}$$

$$\boxed{F(s) = \frac{A}{s-\alpha-j\beta} + \frac{B}{s-\alpha+j\beta} + \frac{N_1(s)}{D_1(s)}}$$

Here $\frac{N_1(s)}{D_1(s)}$ is the remainder term

$$\boxed{\begin{aligned} A &= F(s)\cdot(s-\alpha-j\beta)\Big|_{s=\alpha+j\beta} \\ B &= F(s)\cdot(s-\alpha+j\beta)\Big|_{s=\alpha-j\beta} \end{aligned}}$$

Ex: Find $\mathcal{L}^{-1} \frac{1}{s(s^2+1)}$

let $F(s) = \frac{1}{s\cdot(s^2+1)} = \frac{A}{(s-j)} + \frac{B}{s+j} + \frac{C}{s}$

$$A = F(s)\cdot(s-j)\Big|_{s=j} = \frac{1}{s\cdot(s+j)} = \frac{1}{-2}$$

$$B = F(s)\cdot(s+j)\Big|_{s=-j} = \frac{1}{s\cdot(s-j)}\Big|_{s=-j} = -\frac{1}{2}$$

$$C = F(s)\cdot s\Big|_{s=0} = \frac{1}{(s^2+1)}\Big|_{s=0} = 1 \quad \text{or} \quad C = \frac{1}{(s-j)(s+j)}\Big|_{s=0} = \frac{1}{-j\cdot j} = 1$$

$$\therefore F(s) = \frac{-\frac{1}{2}}{s-j} + \frac{-\frac{1}{2}}{s+j} + \frac{1}{s} = \frac{1}{s} - \frac{1}{2}\,\frac{1}{s^2+1}$$

$$\mathcal{L}^{-1} F(s) = u(t) - \frac{1}{2}\sin t$$

$$f(t) = u(t) - \frac{1}{2}\sin t$$

partial Fraction of Multiple roots

consider the function,

$$F(s) = \frac{N(s)}{(s-s_0)^n \cdot D_1(s)}$$

Here there is multiple root of degree "n" at $s = s_0$. The partial fraction expansion of $F(s)$ will be,

$$F(s) = \frac{k_0}{(s-s_0)^n} + \frac{k_1}{(s-s_0)^{n-1}} + \frac{k_2}{(s-s_0)^{n-2}} + \cdots + \frac{k_{n-1}}{(s-s_0)} + \frac{N_1(s)}{D_1(s)}$$

Consider $F_1(s) = (s-s_0)^n \cdot F(s)$

so

$$\boxed{k_j = \frac{1}{j!} \frac{d^j}{ds^j} \left[F_1(s) \right] \Big/_{s=s_0}}$$

Ex: Find the Inverse Laplace transform of $F(s) = \dfrac{s-2}{s(s+1)^3}$

<u>solution</u>

$$F(s) = \frac{k_0}{(s+1)^3} + \frac{k_1}{(s+1)^2} + \frac{k_2}{(s+1)} + \frac{A}{s}$$

$$A = F(s) \cdot s \Big/_{s=0} = \frac{s-2}{(s+1)^2} \Big/_{s=0} = -2$$

consider $F_1(s) = (s+1)^3 \cdot F(s) = \dfrac{s-2}{s}$

$$k_j = \frac{1}{j!} \frac{d^j}{ds^j} \left[F_1(s) \right] \Big/_{s=-1}$$

$$k_0 = F_1(s) \Big/_{s=-1} = \frac{s-2}{s} \Big/_{s=-1} = 3$$

$$K_1 = \frac{1}{1!}\frac{d}{ds}\left[F_1(s)\right]\Big|_{s=-1} = \frac{d}{ds}\left[\frac{s-2}{s}\right]\Big|_{s=-1} = \frac{s-(s-2)}{s^2}\Big|_{s=-1} = 2$$

$$K_2 = \frac{1}{2!}\frac{d^2}{ds^2}\left[F_1(s)\right] = \frac{1}{2}\,\frac{-(+2)\cdot 2\cancel{s}}{s^{\cancel{4}\,3}}\Big|_{s=-1} = \frac{1}{2}\,\frac{-4}{s^3}\Big|_{s=-1} = 2$$

$$F(s) = \frac{3}{(s+1)^3} + \frac{2}{(s+1)^2} + \frac{2}{(s+1)} - \frac{2}{s}$$

$$\mathcal{L}^{-1}F(s) = \frac{3}{2}t^2\cdot e^{-t} + 2\,t\cdot e^{-t} + 2\,e^{-t} - 2u(t)$$

$$f(t) = \left(\frac{3}{2}t^2 + 2t + 2\right)e^{-t} - 2u(t)$$

EX: solve the following differential equation

$$\frac{dy}{dx} + 2y = x(t) \qquad y(0) = 1$$

If $x(t) = 2t$

<u>Solution</u>

$$\mathcal{L}\left[\frac{dy}{dx} + 2y\right] = \mathcal{L}[x(t)]$$

$$\mathcal{L}\frac{dy}{dx} + 2\mathcal{L}y = \mathcal{L}[x(t)]$$

$$\mathcal{L}\frac{dy}{dx} = s \cdot Y(s) - y(0) = sY(s) - 1$$

$$\mathcal{L}y = Y(s)$$

$$\mathcal{L}[x(t)] = \mathcal{L}[2t] = 2\mathcal{L}t = \frac{2}{s^2}$$

$$\therefore\ sY(s) - 1 + 2Y(s) = \frac{2}{s^2}$$

$$Y(s)[s+2] - 1 = \frac{2}{s^2} \Rightarrow Y(s)[s+2] = \frac{2}{s^2} + 1$$

$$\therefore\ Y(s)[(s+2)] = \frac{2+s^2}{s^2}$$

$$Y(s) = \frac{s^2+2}{s^2(s+2)} = \frac{k_0}{s^2} + \frac{k_1}{s} + \frac{A}{s+2}.$$

$$Y_1(s) = Y(s) \cdot s^2 \Big/ = \frac{s^2+2}{s+2}$$

$$k_0 = Y_1(s)\Big|_{s=0} = \frac{s^2+2}{s+2} = 1$$

$$k_1 = \frac{d}{ds}[Y_1(s)]\Big|_{s=0} = \frac{(s+2)\cdot 2s - (s^2+2)}{(s+2)^2}\Big|_{s=0} = -\frac{1}{2}$$

$$A = Y(s)\cdot(s+2)\Big|_{s=-2} = \frac{s^2+2}{s^2}\Big|_{s=-2} = \frac{6}{4} = \frac{3}{2}$$

$$Y(s) = \frac{1}{s^2} - \frac{1/2}{s} + \frac{3/2}{s+2}$$

$$y(t) = \mathcal{L}^{-1} Y(s) = t^2 - \frac{1}{2} u(t) + 3/2\, e^{-2t}$$

Ex: solve

$$y'' - y = e^{-3t} \qquad \text{if} \quad y(0) = 1 \,, \quad \dot{y}(0) = 2$$

<u>Solution</u>

$$\mathcal{L}[y'' - y] = \mathcal{L}[e^{3t}]$$

$$\mathcal{L} y'' - \mathcal{L} y = \mathcal{L}[e^{-3t}]$$

$$\mathcal{L} y'' = s^2 Y(s) - s\, y(0) - \dot{y}(0) = s^2 Y(s) - s - 2$$

$$\mathcal{L} y = Y(s)$$

$$\mathcal{L} e^{3t} = \frac{1}{s+3}$$

$$s^2 Y(s) - s - 2 - Y(s) = \frac{1}{s+3}$$

$$Y(s)[(s^2-1)] - (s+2) = \frac{1}{s+3}$$

$$Y(s)[s^2-1] = \frac{1}{s+3} + (s+2) = \frac{1 + (s+3)(s+2)}{(s+3)} = \frac{1 + s^2 + 5s + 6}{s+3}$$

$$Y(s)[s^2-1] = \frac{s^2+5s+7}{s+3} \Rightarrow Y(s) = \frac{s^2+5s+7}{(s+3)(s^2-1)}$$

$$Y(s) = \frac{s^2+5s+7}{(s+3)(s-1)(s+1)} = \frac{A}{(s+3)} + \frac{B}{(s-1)} + \frac{C}{s+1}$$

$$A = Y(s) \cdot (s+3) \Big/_{s=-3} = \frac{s^2+5s+7}{(s^2-1)} \Big/_{s=-3} = \frac{1}{8}$$

$$B = Y(s) \cdot (s-1) \Big/_{s=1} = \frac{s^2+5s+7}{(s+3)(s+1)} \Big/_{s=1} = \frac{13}{8}$$

$$c = Y(s)\cdot(s+1)\Big|_{s=-1} = \frac{s^2+5s+7}{(s+3)(s-1)}\Big|_{s=-1} = \frac{3}{-4} = -3/4$$

$$Y(s) = \frac{1/8}{s+3} + \frac{13/8}{s-1} - \frac{3/4}{s+1}$$

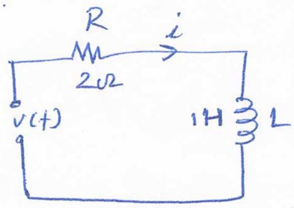
$$\mathcal{L}^{-1}\, Y(s) = y(t)$$

$$\therefore y(t) = \frac{1}{8}e^{-3t} + \frac{13}{8}e^{+t} - 3/4\, e^{-t}$$

EX: For the cct shown in figure below:

1- Derive the differential equation of the cct

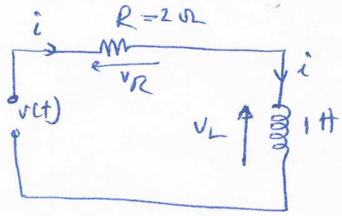
2- Solve the above differential equation if $v(t) = t$ and $i(0) = 1$



Solution $V_L = Z_L \cdot i$

$$V_L = jWL\cdot i = S\cdot L\cdot i = L\cdot Si$$

$$V_L = L\cdot\frac{di}{dt} = 1\cdot i'$$



$$V(t) - V_R - V_L = 0$$

$$V(t) = V_R + V_L \Rightarrow V_R + V_L = V(t)$$

$$i\cdot R + L\cdot i' = t$$

$$i' + 2i = t$$

$$\mathcal{L}[i' + 2i] = \mathcal{L}[t] \Rightarrow \mathcal{L}i' + 2\mathcal{L}i = \mathcal{L}(t)$$

$$s\,I(s) - i(0) + 2\,I(s) = \frac{1}{s^2}$$

$$s\,I(s) - 1 + 2\,I(s) = \frac{1}{s^2} \Rightarrow I(s)[s+2] - 1 = \frac{1}{s^2}$$

$$I(s)[s+2] = \frac{1}{s^2} + 1 = \frac{1+s^2}{s^2}$$

$$I(s) = \frac{s^2+1}{s^2(s+2)}$$

$$I(s) = \frac{A}{(s+2)} + \frac{k_0}{s^2} + \frac{k_1}{s} \quad , \quad I_1(s) = I(s)\cdot s^2 = \frac{s^2+1}{(s+2)}$$

$$A = I(s)\cdot(s+2)\Big|_{s=-2} = \frac{s^2+1}{s^2}\Big|_{s=-2} = \frac{5}{4}$$

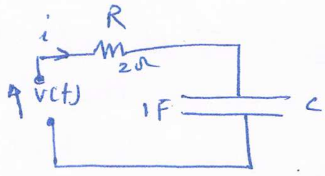
$$k_0 = I_1(s)\Big|_{s=0} = \frac{s^2+1}{s+2}\Big|_{s=0} = \frac{1}{2}$$

$$k_1 = \frac{d}{ds}\left[I_1(s)\right]\Big|_{s=0} = \frac{(s+2)\cdot 2s - (s^2+1)}{(s+2)^2}\Big|_{s=0} = -\frac{1}{4}$$

$$I(s) = \frac{5}{4}\,\frac{1}{s+2} + \frac{1}{2}\,\frac{1}{s^2} - \frac{1}{4}\,\frac{1}{s}$$

$$i(t) = \mathcal{L}^{-1} I(s) = \frac{5}{4} e^{-2t} + \frac{1}{2} t - \frac{1}{4} u(t)$$

EX: Derive an expression of the following cct, solve the above differential equation if $v(t) = \sinh t$



<u>Solution</u>

$$i = \frac{V_C}{Z_C} = \frac{V_C}{\frac{1}{sC}} = C\cdot sV_C$$

$$i = C\cdot\frac{dV_C}{dt} = C\cdot\dot{V}_C$$

$$V_R + V_C = v(t) \Longrightarrow iR + V_C = v(t)$$

$$C\cdot\dot{V}_C\cdot R + V_C = v(t) \Longrightarrow CR\cdot\dot{V}_C + V_C = v(t)$$

$$RC\,\mathcal{L}\dot{V}_C + \mathcal{L}V_C = \mathcal{L}v[t]$$

$$RC\left[S\,V_C(s) - V_C(0)\right] + V_C(s) = \frac{1}{s^2-1}$$

$$2S\,V_C(s) + V_C(s) = \frac{1}{s^2-1}$$

$$V_C(s)\left[2s+1\right] = \frac{1}{s^2-1} \Longrightarrow V_C(s) = \frac{1}{(s^2-1)(2s+1)}$$

$$V_c(s) = \frac{1/2}{(s^2-1)(s+\frac{1}{2})} = \frac{1/2}{(s-1)(s+1)(s+1/2)} = \frac{A}{(s-1)} + \frac{B}{(s+1)} + \frac{C}{s+1/2}$$

$$A = V_c(s)\cdot(s-1)\Big|_{s=1} = \frac{1/2}{(s+1)(s+1/2)}\Big|_{s=1} = \frac{1}{6}$$

$$B = V_c(s)\cdot(s+1)\Big|_{s=-1} = \frac{1/2}{(s-1)(s+1/2)}\Big|_{s=-1} = 1/2$$

$$C = V_c(s)\cdot(s+\tfrac{1}{2})\Big|_{s=-\frac{1}{2}} = \frac{1/2}{s^2-1}\Big|_{s=-\frac{1}{2}} = -\frac{2}{3}$$

$$V_c(s) = \frac{1}{6}\frac{1}{(s-1)} + \frac{1}{2}\frac{1}{(s+1)} - \frac{2}{3}\frac{1}{s+1/2}$$

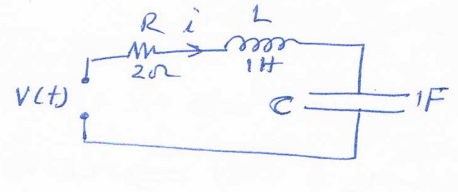
$$V_c(t) = \mathcal{L}^{-1} V_c(s) = \frac{1}{6}e^{t} + \frac{1}{2}e^{-t} - \frac{2}{3}e^{-\frac{1}{2}t}$$

$$\therefore V_c(t) = \frac{1}{6}e^{t} + \frac{1}{2}e^{-t} - \frac{2}{3}e^{-0.5t}$$

<u>H.W</u>: For the cct shown below

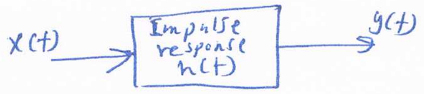
1- Derive the differential equation of it

2- solve the above differential equation if $v(t) = t \cdot e^{-t}$



system Transfer function

we know that the output of LTI CT system is given as;



$$y(t) = h(t) * x(t)$$

Taking the L.T. of above equation

$$\mathcal{L}\, y(t) = \mathcal{L}\,[h(t) * x(t)]$$

$$Y(s) = H(s) \cdot X(s)$$

where $X(s)$ is Laplace Trans. of input signal
$Y(s)$ = = = = output =
$H(s)$ = = = = impulse response

$H(s)$ also is called the transfer function $\left(H(s) = \dfrac{Y(s)}{X(s)} \right)$

and

$$h(t) = \mathcal{L}^{-1}\, H(s)$$

Frequency response of the transfer function ($H(s)$) is given by

$$H(s)\Big|_{s=j\omega}$$

Ex: The input-output relation of a system is given by,

$$\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 3y(t) = \frac{dx(t)}{dt} + 2x(t)$$

Find the transfer function, frequency response and impulse response.

<u>solution</u>

$$s^2 Y(s) + 4sY(s) + 3Y(s) = s\,X(s) + 2X(s)$$

$$Y(s)\,(s^2+4s+3) = X(s)\,(s+2)$$

$$\frac{Y(s)}{X(s)} = \frac{(s+2)}{s^2+4s+3}$$

$$H(s) = \frac{s+2}{s^2+4s+3} \quad \text{transfer function}$$

$$\text{frequency response} = H(s)\Big|_{s=j\omega}$$

$$H(j\omega) = \frac{j\omega+2}{(j\omega)^2+4j\omega+3} = \frac{2+j\omega}{(3-\omega^2)+4j\omega}$$

$$h(t) = \mathcal{L}^{-1} H(s) \quad \text{impulse response}$$

$$H(s) = \frac{s+2}{(s+1)(s+3)} = \frac{A}{s+1} + \frac{B}{s+3}$$

$$A = H(s)\cdot(s+1)\Big|_{s=-1} = \frac{s+2}{s+3}\Big|_{s=-1} = \frac{1}{2}$$

$$B = H(s)\cdot(s+3)\Big|_{s=-3} = \frac{s+2}{s+1}\Big|_{s=-3} = \frac{1}{2}$$

$$H(s) = \frac{1/2}{s+1} + \frac{1/2}{(s+3)}$$

$$h(t) = \mathcal{L}^{-1} H(s) = \frac{1}{2}e^{-t} + \frac{1}{2}e^{-3t}$$

Ex: The transfer function of system is given by

$$H(s) = \frac{s}{s^2+5s+6}$$

The input to the system is $x(t) = e^{-t}u(t)$. Determine the output assuming zero initial conditions

Solution

$$H(s) = \frac{Y(s)}{X(s)}$$

$$\therefore Y(s) = H(s) \cdot X(s)$$

$$X(s) = \mathcal{L}\, x(t) = \frac{1}{s+1}$$

$$Y(s) = H(s) \cdot X(s) = \frac{s}{s^2+5s+6} \cdot \frac{1}{s+1}$$

$$Y(s) = \frac{s}{(s+2)(s+3)(s+1)} = \frac{A}{s+2} + \frac{B}{s+3} + \frac{C}{s+1}$$

$$A = Y(s) \cdot (s+2) \Big/_{s=-2} = \frac{s}{(s+3)(s+1)} \Big/_{s=-2} = 2$$

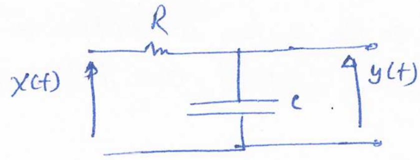
$$B = Y(s) \cdot (s+3) \Big/_{s=-3} = \frac{s}{(s+2)(s+1)} \Big/_{s=-3} = \frac{-3}{2}$$

$$C = Y(s) \cdot (s+1) \Big/_{s=-1} = \frac{s}{(s+2)(s+3)} \Big/_{s=-1} = -\frac{1}{2}$$

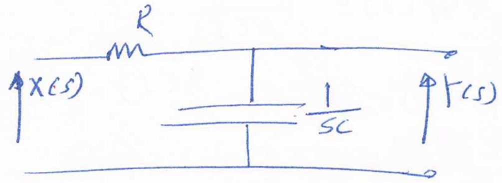
$$Y(s) = \frac{2}{s+2} - \frac{3/2}{s+3} - \frac{1/2}{s+1}$$

$$y(t) = \mathcal{L}^{-1}\, Y(s) = \left[2e^{-2t} - \frac{3}{2}e^{-3t} - \frac{1}{2}e^{-t}\right] u(t)$$

Ex: Derive the transfer function of the system shown in fig. below and find the impulse response



Solution



using voltage divider

$$Y(s) = X(s)\,\frac{\frac{1}{sC}}{R+\frac{1}{sC}} = \frac{1}{sRC+1} \cdot X(s)$$

$$\frac{Y(s)}{X(s)} = \frac{1}{sRC+1} = \frac{1/RC}{s+1/RC}$$

$$\therefore H(s) = \frac{1/RC}{s+1/RC}$$

$$h(t) = \mathcal{L}^{-1} H(s) = \frac{1}{RC} e^{-\frac{1}{RC}t}$$

H.W The differential equation of the system is given as,

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = x(t) \qquad \text{with } y(0) = 3 \text{ and } y'(0) = -5$$

Find the output for $x(t) = 2u(t)$

ans: $y(t) = [1 - e^{-t} + 3e^{-2t}]\, u(t)$